Divisibility Rules

It's my sincere hope that one of your elementary school teachers taught you these rules. If not, we can fix that. The rules can be broken down into 5 groups:

- 2, 4, and 8: Powers of 2
- 3 and 9: Multiples of 3
- 5 and 10: Multiples of 5
- 6: The mother of all divisibility rules

However we can use the rules below to construct rules for lots of composite numbers. I'll point out a couple of examples as we go along.

Divisibility Rules for 2, 4, and 8

1) A number is divisible by 2 if
   a) **The last digit is divisible by 2.** For example 2,106 is divisible by two because the last digit, 6, is divisible by 2, but 2,119 is not divisible by 2 because 9 is not divisible by two.
2) A number is divisible by 4 if
   a) **The last two digits form a number that is divisible by 4.** For example 3,436 is divisible by 4 because 36 is divisible by 4, but 3,774 is not divisible by 4 because 74 is not divisibly by 4. Note that if the digit in the ten's place is a zero the rule still applies; 104 and 2,308 are both divisible by 4.
   b) **It is divisible by 2 twice.** For example, $256 \div 2 = 128$, and $128 \div 2 = 64$, so 256 is divisible by 4, but 170 is not divisible by 4 because $170 \div 2 = 85$, and 85 is not divisible by 2. Thus, 170 is only divisible by 2 once, so it is not divisible by 4.
3) A number is divisible by 8 if
   a) **The last three digits form a number that is divisible by 8.** For example 45,856 is divisible by 8 because 856 is divisible by 8, but 45,764 is not divisible by 8 because 764 is not divisible by 8. Again, if zeros appear in the hundreds or tens place, the rule still applies; 45,056 and 45,008 are both divisible by 8.
   b) **The number is divisible by 2 three times.** For example 128 is divisible by 8 because $128 \div 2 = 64$, $64 \div 2 = 32$, and $32 \div 2 = 16$. This rule is not as practical as the similar rule for 4, but it will work in a pinch.

There are similar rules for all powers of two – for 2^n, check the the number formed by the last n digits for divisibility by 2^n.

Divisibility Rules for 3 and 9

1) **A number is divisible by 3 if the sum of the digits is divisible by 3.** For example, 432 is divisible by 3 because $4 + 3 + 2 = 9$ and 9 is divisible by 3, but 76,517 is not divisible by 3 because $7 + 6 + 5 + 1 + 7 = 26$, and 26 is not divisible by 3.

2) **A number is divisible by 9 if the sum of the digits is divisible by 9**. For example, 6,588 is divisible by 9 because $6 + 5 + 8 + 8 = 27$ and 27 is divisible by 9, but 751 is not divisible by 9 because $7 + 5 + 1 = 13$, and 13 is not divisible by 9.

Divisibility Rules for 5 and 10

1) **A number is divisible by 5 if it ends in a 5 or a zero**. For, example 5, 25, 125, and 625 are all divisible by 5 because they end in five, and 10, 100, and 650 are divisible by 5 because they end in zero.
2) **A number is divisible by 10 if it ends in zero.** For example 550, 760, 10,000, and 5,000,000 are all divisible by 10 because they end in zero.

There's a rule for 15 that we'll discuss in the next chapter. You should try to figure it out for yourself – think about the progression of the 10's digit.

Divisibility Rule for 6

A number is divisible by 6 if it divisible by 2 and 3. For example, 312 is divisible by 6 because it is divisible by 2 and 3 (if you don't know why 312 is divisible by 2 and 3, see above and apply a ruler to your knuckles).

I've placed this rule last in the list because it can be used to find other rules, but only with a few important caveats.

Many students try to generalize this rule like so: *If a number is divisible by two integers, then it is also divisible by the product of those integers.* It makes sense, right? For example, 12 is divisible by 3 and 4, so any number that is divisible by 3 and 4 is also divisible by 12. We can show this is true by examining the positive multiples of 3 and the multiples of 4:

- Multiples of 3: 3, 6, 9, **12**, 15, 18, 21, **24**, 27, 30, 33, **36**, 39, 42, 45, **48**…
- Multiples of 4: 4, 8, **12**, 16, 20, **24**, 28, 32, **36**, 40, 44, **48**…

The multiples that 3 and 4 have in common are the ones in bold, and they're all divisible by 12.

Unfortunately, this rule has an infinite number of counter-examples. For example, if a number is divisible by 2 and 6 it is **not** necessarily divisible by 12. Six is divisible by 2 and 6, but it is not divisible by 12.

Understanding when and why this works, and when and why it does not work is extremely important. We'll go into detail later, but you might want to think about it for a minute or two now.

Quiz 1.2

1. Fill out the following divisibility table. Indicate whether the number in the column is divisible by the number in the corresponding row by writing a Y for "yes" or an N for "no." If yes, include the quotient. I've written in some answers for illustration.

	56	62	91	36	972	4,374	6,736
Div by 2		Y(31)					
Div by 4			N				
Div by 8							Y(844)
	81	**924**	**657**	**579**	**7,489**	**6,565**	**6,123**
Div by 3							
Div by 9							
	65	**135**	**270**	**4555**	**10,000**	**67,895**	$5^4 2^4$
Div by 5							
Div by 10							
	272	**657**	**720**	**972**	**6,735**	**9,240**	$2^3 3^5$
Div by 6							

2. True or false? If the statement is true, explain why. If it is false, provide a counterexample.

a) If an integer is divisible by 3, it is also divisible by 9.
b) If an integer is divisible by 9, it is also divisibly by 3
c) If an integer is divisible by 100, it is also divisible by 25
d) If an integer is divisible by 2 and 6, it is also divisible by 12
e) If an integer is divisible by 3 and 4, it is also divisible by 12

Quiz 1.2 Solutions

1. Fill out the following divisibility table. Indicate whether the number in the column is divisible by the number in the corresponding row by writing a Y for "yes" or an N for "no." If yes, include the quotient. I've written in some answers for illustration.

	56	**62**	**91**	**36**	**972**	**4,374**	**6,736**
Div by 2	Y (28)	Y(31)	N	Y(18)	Y(486)	Y(2,187)	Y(3,368)
Div by 4	Y(14)	N	N	Y(9)	Y (243)	N	Y(1,688)
Div by 8	Y(7)	N	N	N	N	N	Y(844)
	81	**924**	**657**	**579**	**7,489**	**6,565**	**6,123**
Div by 3	Y(27)	Y(308)	Y(219)	Y(193)	N	N	Y(2,041)
Div by 9	Y(9)	N	Y(73)	N	N	N	N
	65	**135**	**270**	**4555**	**10,000**	**67,895**	**$5^4 2^4$**
Div by 5	Y(13)	Y(27)	Y(54)	Y(911)	Y(2,000)	Y(13,579)	Y($5^3 2^4$)
Div by 10	N	N	Y(27)	N	Y(1,000)	N	Y($5^3 2^3$)
	272	**657**	**720**	**972**	**6,735**	**9,240**	**$2^3 3^5$**
Div by 6	N	N	Y(120)	Y(162)	N	Y(1,540)	Y($2^2 3^4$)

2. True or false? If the statement is true, explain why. If it is false, provide a counterexample.

a) If an integer is divisible by 3, it is also divisible by 9.
   FALSE: Twelve is divisible by 3, but it is not divisible by 9.
b) If an integer is divisible by 9, it is also divisibly by 3
   TRUE: If an integer is divisible by 9, it is also divisible by any factors of 9. Since 3 is a factor of 9, any integer which is divisible by 9 is also divisible by 3.
c) If an integer is divisible by 100, it is also divisible by 25
   TRUE: If an integer is divisible by 100, it is also divisible by any factors of 100. Since 25 is a factor of 100, any integer which is divisible by 100 is also divisible by 25.
d) If an integer is divisible by 2 and 6, it is also divisible by 12

FALSE: Six is divisible by 2 and 6, but it is not divisible by 12. This rule fails because 2 and 6 share a factor, 2.

e) If an integer is divisible by 3 and 4, it is also divisible by 12
TRUE: Because 3 and 4 have no common factors (other than 1, which doesn't count in this situation), both of them have to be factors of an integer divisible by 3 and 4. Consequently, 12 is also a factor.